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On the L_p-Theorems for Index Transforms

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Abstract

This paper is devoted to study index transforms by general constructions of kernels, which involve the known Kontorovich-Lebedev and the Mehler-Fock integral transfroms, and the index transforms with Meijer's G-function and Fox's H-function as kernels. Using the Mellin-Parseval equality, the general index transform can be written through Mellin images, which produces a number of examples. Mapping properties and inversion theorems on the space $L_{\nu,p}(\mathbf{R}_+)$ of functions f with

$$\int_0^\infty |t^\nu f(t)|^p \frac{dt}{t} < \infty \quad (1 \le p \le 2; \nu \in \mathbf{R})$$

are investigated. Several examples of the index transform are considered.

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1. Introduction

The present paper deals with general index transform of the form

$$\left[Y_{i\tau}^{\varphi} f\right](\tau) = \tau \int_0^\infty Y_{i\tau}^{\varphi}(t) f(t) dt \quad (\tau > 0). \tag{1.1}$$

The kernel function $Y_{i\tau}^{\varphi}(x)$ is given by

(1.2)

$$Y_{i\tau}^{\varphi}(x) = \frac{1}{4\pi i} \int_{1-\nu-i\infty}^{1-\nu+i\infty} \Gamma\left(\frac{1-s+i\tau}{2}\right) \Gamma\left(\frac{1-s-i\tau}{2}\right) \varphi^*(s)(2x)^{-s} ds \quad (x > 0, \nu > 0)$$

involving the Euler gamma-functions and $\varphi^*(s)$ is an arbitrary function such that the convergence of the integral (1.2) is meant at least in the norm of $L_{\nu,p}$ $(p > 1)$. The formula (1.2) is very close to the known Mellin-Parseval equality (see below). For our further investigations we need to present some elements of the theory of the Mellin transform [6].

Let $L_{\nu,p}(\mathbf{R}_+)$ be the space of functions equipped by the norm

$$\|f\|_{\nu,p} = \left(\int_0^\infty |t^\nu f(t)|^p \frac{dt}{t}\right)^{1/p} < \infty \tag{1.3}$$

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with $p \geq 1$ and $\nu \in \mathbf{R}$, where $\mathbf{R} = (-\infty, \infty)$ and $\mathbf{R}_+ = (0, \infty)$. Note that $L_{1/p,p}(\mathbf{R}_+) \equiv L_p(\mathbf{R}_+)$. For $f \in L_{\nu,p}(\mathbf{R}_+)$ with $1 < p \leq 2$ the Mellin transform is defined by [6]

$$(1.4) \qquad f^*(s) = \int_0^\infty f(t)t^{s-1}dt \quad (\mathrm{Re}(s) = \nu),$$

where the convergence of the integral (1.4) is in the mean by the norm of the space $L_q(\nu - i\infty, \nu + i\infty)$ for $q = p/(p-1)$, namely

$$(1.5) \qquad \lim_{N \to \infty} \left\| f^*(s) - \int_{1/N}^N f(t)t^{s-1}dt \right\|_{L_q(\nu - i\infty, \nu + i\infty)} = 0$$

and

$$(1.6) \qquad \|f^*(s)\|_{L_q(\nu - i\infty, \nu + i\infty)} = \frac{1}{2\pi} \left(\int_{\nu - i\infty}^{\nu + i\infty} |f^*(s)|^q ds \right)^{1/q}.$$

In particular, if $f \in L_{\nu,p}(\mathbf{R}_+) \cap L_{\nu,1}(\mathbf{R}_+)$, the integral (1.4) is the usual improper absolutely convergent integral.

In the following discussions we fix parameters as $1 < p \leq 2, q = p/(p-1)$ and $\nu \in \mathbf{R}_+$, unless otherwise stated.

Let us give some useful results from [6].

Theorem 1. *If $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, then its Mellin transform $f^*(s) \equiv f^*(\nu + it)$ exists and belongs to the space $L_q(\mathbf{R})$.*

Theorem 2. *If $f^*(\nu + it) \in L_p(\mathbf{R})$, then the inverse Mellin transform*

$$(1.7) \qquad f(x) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} f^*(s)x^{-s}ds \quad (x > 0)$$

exists with its convergence in the mean in $L_{\nu,q}$ and $f(x) \in L_{\nu,q}(\mathbf{R}_+)$. Moreover, the equality

$$(1.8) \qquad f(x) = \frac{1}{2\pi i} \frac{d}{dx} \int_{\nu - i\infty}^{\nu + i\infty} \frac{f^*(s)}{1 - s} x^{1-s} ds \quad (x > 0)$$

is true almost everywhere on $\mathbf{R}_+$.

Theorem 3. *If $f^*(\nu + it) \in L_p(\mathbf{R})$ and $h(x) \in L_{1-\nu,p}(\mathbf{R}_+)$, then the Mellin-Parseval equality takes place*

$$(1.9) \qquad \int_0^\infty f(xt)h(t)dt = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} f^*(s)h^*(1 - s)x^{-s}ds.$$

In order to let the integrand in the representation (1.2) satisfy the assumption of Theorem 2, we use the asymptotic by the Stirling formula for the gamma-function [1] when $s \in (1 - \nu - i\infty, 1 - \nu + i\infty)$. Thus for each $\tau > 0$ and for $s = 1 - \nu + it$ we obtain

$$(1.10) \qquad \Gamma\left(\frac{1 - s + i\tau}{2}\right) \Gamma\left(\frac{1 - s - i\tau}{2}\right) = O\left(e^{-\pi|t|/2} |t|^{\nu - 1}\right) \quad (|t| \to \infty).$$

Hence if $\varphi^*(1-\nu-it)e^{-\pi|t|/2}|t|^{\nu-1} \in L_p(-\infty,\infty)$, then by Theorem 2 we obtain that $Y_{i\tau}^{\varphi}(x) \in L_{1-\nu,q}(\mathbf{R}_+)$. Moreover, if, in addition, there exists such a function $\varphi(x) \in L_{1-\nu,p}(\mathbf{R}_+)$ being the inverse Mellin transform of the function $\varphi^*(s)$ that $\varphi^*(1-\nu+it) \in L_q(\mathbf{R})$, then invoking to the Mellin-Barnes integral representation [4, §10, 9.3(1)] for the Macdonald function $K_{i\tau}(x)$ [1]

$$K_{i\tau}(x) = \frac{1}{4\pi i}\int_{\nu-i\infty}^{\nu+i\infty} 2^{s-1}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)x^{-s}ds \quad (x>0), \tag{1.11}$$

we can apply Theorem 3 to establish the formula for the kernel (1.2)

$$Y_{i\tau}^{\varphi}(x) = \int_0^\infty K_{i\tau}(y)\varphi(xy)dy \quad (x>0). \tag{1.12}$$

From the asymptotic behavior [1] $K_{i\tau}(x) = O(\log x)$ $(x \to 0+)$, $K_{i\tau}(x) = O(e^{-x}/\sqrt{x})$ $(x \to +\infty)$, it follows that $K_{i\tau}(x) \in L_{\nu,q}(\mathbf{R}_+)$. Meanwhile as we have seen above, the integral (1.2) admits such functions $\varphi^*(s)$ that can not belong to any space $L_p(\nu-\infty, \nu+\infty)$ and the respective integral (1.7) diverges in the mean sense, too. For instance, if we take $\varphi^*(s) = [\Gamma(s/2)]^{-1}$, then our assumptions are true and, however, the integral (1.2) is absolutely convergent as it is easily seen from the Stirling formula. Consequently, the index kernel $Y_{i\tau}^{\varphi}(x)$ exists. Keeping the sign $*$ of the Mellin transform for the notation $\varphi^*(s)$ we can extend a number of examples of kernels given by the general formula (1.2).

The Macdonald function mentioned above by the formulas (1.11) and (1.12) with an imaginary index is the kernel of the familiar Kontorovich-Lebedev transform pair [2]

$$[K_{i\tau}f] \equiv g(\tau) = \tau\int_0^\infty K_{i\tau}(y)f(y)dy \quad (\tau>0), \tag{1.13}$$

$$xf(x) = \frac{2}{\pi^2}\int_0^\infty \sinh(\pi\tau)K_{i\tau}(x)g(\tau)d\tau \quad (x>0). \tag{1.14}$$

More precise speaking, the index transform (1.1) generalizes the formula (1.13) for the direct Kontorovich-Lebedev transform as it is not difficult to conclude by putting in (1.2) $\varphi^*(s) \equiv 1$ and by appealing to the Mellin-Barnes integral (1.11).

It is well-known that the Macdonald function has the expression [1]

$$K_{i\tau}(x) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x\cosh\beta}e^{i\tau\beta}d\beta \quad (x>0). \tag{1.15}$$

By the analytic property of the integrand in (1.15) and by its asymptotic behavior at the contour we can shift it along the horizontal open infinite strip $(i\delta-\infty, i\delta+\infty)$ with $\delta \in [0, \pi/2)$ as

$$K_{i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta}e^{i\tau\beta}d\beta \quad (x>0). \tag{1.16}$$

Note here the useful uniform estimate of the Macdonald function [10]

$$|K_{i\tau}(x)| \le C_\delta\frac{\tau+1}{\tau}e^{-\delta\tau - x\cos\delta} \quad (\tau, x>0), \tag{1.17}$$

where $0 < \delta < \pi/2$ and C_δ is a positive constant depending only on δ. In [8] first and later in [9], [7] it was constructed a generalization of the Kontorovich-Lebedev index transform (1.13)-(1.14) on the case of the Meijer G-function [1] as the kernel. As it was shown ([10]), this general index transform comprises enough wide class of integral transforms such as the Mehler-Fock transform [11], [13], the Olevskii transform [10], the Lebedev-Skalskaya transforms [12] and its new generalizations. Detailed information about index transforms and modern results in this field can be found in the book [10].

In this paper we continue to extend a number of examples of index transforms in L_p. Some theorems are proved recently at our previous paper [14] in the case when the general kernel (1.2) has the integral representation (1.12).

Finally, let us note the Hölder inequality for the weighted spaces

$$\int_0^\infty |f(t)h(t)|dt \le \|f\|_{\nu,p}\|h\|_{1-\nu,q}, \tag{1.18}$$

where $f(x) \in L_{\nu,p}(\mathbf{R}_+), h(x) \in L_{1-\nu,q}(\mathbf{R}_+)$ and the generalized Minkowski inequality

$$\left(\int_0^\infty dx \left|\int_0^\infty f(x,y)dy\right|^p\right)^{1/p} \le \int_0^\infty dy \left(\int_0^\infty |f(x,y)|^p dx\right)^{1/p} \tag{1.19}$$

for $p > 1$.

2. General Results

In this section we will investigate the general index transform (1.1) in the space $L_{\nu,p}(\mathbf{R}_+)$ and will establish its inversion theorem in this space.

Theorem 4. *Let $f(x) \in L_{\nu,p}(\mathbf{R}_+)$ and $\varphi^*(1-\nu+it)e^{-\pi|t|/2}|t|^{\nu-1} \in L_p(-\infty,\infty)$. If $\varphi^*(s)f^*(1-s) \in L_p(1-\nu-i\infty, 1-\nu+i\infty)$, then the general index transform* (1.1) *can be represented by the formula*

$$[Y_{i\tau}^{\varphi} f] = \tau \int_0^\infty K_{i\tau}(y)(\Phi f)(y)dy \quad (\tau > 0), \tag{2.1}$$

where the operator $(\Phi f)(x)$ is defined by the integral

$$(\Phi f)(x) = \frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} \varphi^*(s)f^*(1-s)x^{-s}ds \quad (x > 0) \tag{2.2}$$

with the convergence in the mean by the norm $L_{1-\nu,q}$.

Proof. This theorem can be proved by using the Mellin-Parseval equality (1.9) and Theorem 3. We start from the condition $f(x) \in L_{\nu,p}(\mathbf{R}_+)$. By invoking to Theorem 3 and by rewriting (1.1) by the right hand-side of (1.9), it is enough to have for each $\tau > 0$ the Mellin transform of the kernel $Y_{i\tau}^{\varphi}(x)$ from the space $L_p(1-\nu-i\infty, 1-\nu+i\infty)$. However, from the formula (1.2) we conclude that if $\varphi^*(1-\nu+it)e^{-\pi|t|/2}|t|^{\nu-1} \in L_p(-\infty,\infty)$, then one can achieve the property that the integrand in (1.2) belongs to the space $L_p(1-\nu-i\infty, 1-\nu+i\infty)$. Moreover, by Theorem 2 it follows that the index kernel (1.2) is from the space $L_{1-\nu,q}(\mathbf{R}_+)$. The Hölder inequality (1.18) immediately implies that the integral

(1.1) is absolutely convergent under the above assumptions. So by the Mellin-Parseval equality (1.9) we reduce (1.1) to the relation

$$(2.3)\quad [Y_{i\tau}^{\varphi}f](\tau)=\frac{\tau}{4\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty}2^{-s}\Gamma\left(\frac{1-s+i\tau}{2}\right)\Gamma\left(\frac{1-s-i\tau}{2}\right)\varphi^*(s)f^*(1-s)ds \quad (\nu>0).$$

Further it is clear now that we need to back from the right-hand side of (1.9) to the left-hand side in order to establish the representation (2.1). It is possible, for instance, if $\varphi^*(s)f^*(1-s)\in L_p(1-\nu-i\infty,1-\nu+i\infty)$. As is obvious from the asymptotic behavior of the Macdonald function $K_{i\tau}(x)$ by the variable x (see above), it belongs to $L_{\nu,p}(\mathbf{R}_+)$. Hence we arrive at (2.1) with the integral operator $(\Phi f)(x)$ defined by (2.2) and belonging to $L_{1-\nu,q}(\mathbf{R}_+)$, as it follows from Theorem 2. Thus Theorem 4 is proved.

Theorem 5. *Under conditions of the previous theorem the general index transform is a bounded operator into the space $L_r(\mathbf{R}_+)$ $(r\geq 1)$ and there holds the estimate*

$$(2.4)\qquad \|\,[Y_{i\tau}^{\varphi}f]\,\|_{L_r}\leq C\|(\Phi f)\|_{1-\nu,q}<\infty \quad (q>1),$$

where C is a positive constant.

Proof. In view of the estimate (1.17) and by the Hölder inequality (1.18), we have

$$(2.5)\quad |[Y_{i\tau}^{\varphi}f](\tau)|\leq\tau\int_0^\infty|K_{i\tau}(t)(\Phi f)(t)|\,dt\leq C_\delta(\tau+1)e^{-\delta\tau}\int_0^\infty e^{-t\cos\delta}\,|(\Phi f)(t)|\,dt$$
$$\leq C_\delta(\Gamma(\nu p))^{1/p}(p\cos\delta)^{-\nu}\|(\Phi f)\|_{1-\nu,q}(\tau+1)e^{-\delta\tau},$$

for $0<\delta<\pi/2$. Hence we obtain

$$(2.6)\quad \|\,[Y_{i\tau}^{\varphi}f]\,\|_{L_r}=\left(\int_0^\infty|[Y_{i\tau}^{\varphi}f](\tau)|^r\,d\tau\right)^{1/r}$$
$$\leq C_\delta(\Gamma(\nu p))^{1/p}(p\cos\delta)^{-\nu}\|(\Phi f)\|_{1-\nu,q}\left(\int_0^\infty(\tau+1)^re^{-\delta r\tau}d\tau\right)^{1/r}$$
$$=C\|(\Phi f)\|_{1-\nu,q}.$$

This completes the proof of Theorem 5.

Let us now consider the operator

$$(2.7)\qquad \left(I_\varepsilon^{\psi}g\right)(x)=\frac{2}{\pi^2}\int_0^\infty\sinh((\pi-\varepsilon)\tau)Y_{i\tau}^{\psi}(x)g(\tau)d\tau\quad(x>0),$$

where $\varepsilon\in(0,\pi)$ and $Y_{i\tau}^{\psi}(x)$ is the index kernel of the type (1.2) and the characteristic function $\psi(x)$ is defined by the formula

$$(2.8)\qquad Y_{i\tau}^{\psi}(x)=\int_0^\infty K_{i\tau}(y)\psi(xy)dy\quad(x>0).$$

The following Theorems 6 and 7 can be established on the same line of proofs of Theorems 4 and 5 in [14].

Theorem 6. *Let* $\psi \in L_1(\mathbf{R}_+) \cap L_{\nu+1,1}(\mathbf{R}_+)$ $(\nu > 0)$. *Then for the function* $g(\tau) = [Y_{i\tau}^{\varphi} f](\tau)$ *represented by the general index transform* (2.1) *with* $(\Phi f) \in L_{1-\nu,q}(\mathbf{R}_+)$ $(q > 2)$ *the operator* (2.7) *has the form*

$$(2.9) \qquad \left(I_\varepsilon^{\psi} g\right)(x) = \frac{\sin\varepsilon}{\pi} \int_0^\infty \int_0^\infty \frac{uvK_1\left(\sqrt{u^2+v^2-2uv\cos\varepsilon}\right)}{\sqrt{u^2+v^2-2uv\cos\varepsilon}} \psi(xu)(\Phi f)(v)dudv,$$

where $K_1(z)$ *is the Macdonald function of order* 1.

The inversion of the index transform (1.1) in $L_{\nu,p}$ is given by

Theorem 7. *Let* $0 < \nu < 1$ *and* $g(\tau) = [Y_{i\tau}^{\varphi} f]$ *be under assumptions of Theorem* 4 *for* $f(x) \in L_{\nu,p}(\mathbf{R}_+)$. *Let the characteristic function* $\psi(x)$ *satisfying the assumption of Theorem* 6 *be from the space* $L_{1+\nu,p}(\mathbf{R}_+)$ *and* $(\Phi f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$. *Then the equality*

$$(2.10) \qquad \underset{\varepsilon\to 0+}{\text{l.i.m.}} \left(I_\varepsilon^{\psi} g\right)(x) = \frac{1}{x^2}\int_0^x f(y)dy \quad (x > 0)$$

by the $L_{1+\nu,p}$*-norm is valid if and only if the relation*

$$(2.11) \qquad \psi^*(1+s)\varphi^*(1-s) = \frac{1}{1-s} \quad (\mathrm{Re}(s) = \nu)$$

is fulfilled, where the sign " $*$ " *denotes the Mellin transform* (1.4). *In addition, the limit in* (2.10) *exists almost everywhere on* $\mathbf{R}_+$.

3. Examples of Index Transforms

In the present section we apply general results of the previous section and demonstrate various examples of the index transforms and their L_p-inversions. Some of them are new pairs and can be derived from the general transform (1.1) by using the basic tables of Mellin transforms in [4] and [5, Vol.3].

Example 1. The index transform with Whittaker's function. Let us put in the formula (1.2) $\varphi^*(s) = 2^{s+1}/\Gamma(1-\kappa-s/2)$, where $\kappa \in \mathbf{C}$ being a complex number. Then according to the Mellin transform formula [4, §10, 12.6(4)] the index transform (1.1) takes the form involving the Whittaker function

$$(3.1) \qquad \left[W_{\kappa,i\tau/2} f\right](\tau) = 2\tau \int_0^\infty W_{\kappa,i\tau/2}(1/t^2)e^{-1/2t^2} f(t)dt \quad (\tau > 0),$$

which was first introduced by Wimp [8] in slightly different form (see also [10]) as a particular case of the integral transform with respect to an index of the Meijer G-function. The main result for the transform (3.1) is contained in Theorem 7 as the following:

Theorem 8. *Let* $0 < \nu < \min(1 - 2\mathrm{Re}(\kappa), 1)$ *and* $f(x) \in L_{\nu,p}(\mathbf{R}_+)$. *Then under the condition* $(\Phi f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$ *for the operator* (2.2) *the inversion formula for the transform* (3.1) *takes place*

$$\int_0^x f(y)dy = \underset{\varepsilon\to 0+}{\mathrm{l.i.m.}} \frac{1}{4\pi^2}\int_0^\infty\int_0^x ye^{1/2y^2}\sinh((\pi-\varepsilon)\tau)\Gamma\left(\frac{1-i\tau}{2}-\kappa\right) \times \Gamma\left(\frac{1+i\tau}{2}-\kappa\right) W_{\kappa,i\tau/2}\left(\frac{1}{y^2}\right) g(\tau)dyd\tau, \tag{3.2}$$

where $g(\tau) = \left[W_{\kappa,i\tau/2}f\right](\tau)$ *is a bounded operator in any space* $L_r(\mathbf{R}_+)$ $(r \geq 1)$ *and the limit in* (3.2) *is meant in the* $L_{\nu-1,p}$*-norm. Besides, the limit in* (3.2) *exists almost everywhere on* $\mathbf{R}_+$.

Proof. It is easy to check all assumptions of Theorem 4 for the transform (3.1). From the algebraic equality (2.11) we can find the value of the Mellin transform for the characteristic function $\psi(x)$ on the formula (2.8). Furthemore, according to the inversion formula (1.7) for the Mellin transform we have the equality

$$\psi(x) = \frac{1}{2\pi i}\int_{1+\nu-i\infty}^{1+\nu+i\infty} \frac{2^{s-3}\Gamma(s/2-\kappa)}{2-s}x^{-s}ds \quad (x>0). \tag{3.3}$$

Evidently, the integral (3.3) is absolutely and uniformly convergent by $x > 0$ owing to the Stirling formula. Moreover from analytic properties of the integrand in (3.3) being considered as a function of the complex variable s by shifting of the contour, we have $\psi(x) \in L_{1+\nu,1}(\mathbf{R}_+)$. Precisely, there exists the parameter $\delta > 0$ such that $\psi(x) = O(x^{\delta-1})$ $(x \to 0+)$, $\psi(x) = O(x^{-\delta-1})$ $(x \to \infty)$. The property $\psi(x) \in L_{1+\nu,p}(\mathbf{R}_+)$ follows from Theorem 2. If we evaluate the kernel $Y_{i\tau}^{\psi}(x)$ by the substitution (3.3) in the integral like (1.2) for the function $\psi^*(s)$ and invoke to the Mellin transform formula [4, §10, 12.7(4)], we arrive at (3.2). Theorem 8 is completely proved.

Setting $\kappa = 0$ in formulas (3.1) and (3.2) owing to relations [4, §10, 9.13(4) and 9.14(4)] we immediately deduce a modification of the Kontorovich-Lebedev transform pair in $L_{\nu,p}$ $(0 < \nu < 1)$, which can be reduced to (1.13), (1.14). We find here that

$$g(\tau) = \frac{\tau}{\sqrt{\pi}}\int_0^\infty K_{i\tau/2}\left(\frac{1}{2t^2}\right)e^{-1/(2t^2)}f(t)\frac{dt}{t} \quad (\tau > 0), \tag{3.4}$$

$$\int_0^x f(y)dy = \underset{\varepsilon\to 0+}{\mathrm{l.i.m.}} \frac{1}{2\pi\sqrt{\pi}}\int_0^\infty\int_0^x e^{1/(2y^2)}\frac{\sinh((\pi-\varepsilon)\tau)}{\cosh(\pi\tau/2)}K_{i\tau/2}\left(\frac{1}{2y^2}\right)g(\tau)dyd\tau. \tag{3.5}$$

Example 2. The index transform with the square of the Macdonald function. Let us consider the index transform (1.1) by putting $\varphi^*(s) = 2^{s+1}\Gamma((1-s)/2)/\Gamma(1-s/2)$ in (1.2). Making use of the formula [4, §4, 9.37(4)], we obtain the index transform with respect a square of the Macdonald function which was first introduced by Lebedev [3] and investigated by the first author in [10]:

$$g(\tau) = \frac{2\tau}{\sqrt{\pi}}\int_0^\infty K_{i\tau/2}^2\left(\frac{1}{t}\right)f(t)\frac{dt}{t} \quad (\tau > 0). \tag{3.6}$$

Similarly we establish an L_p-theorem for the transform (3.6) evaluating the inversion kernel $Y^{\psi}_{i\tau}(x)$ by the Mellin-Barnes integral

$$(3.7)\ Y^{\psi}_{i\tau}(x)=\frac{2^{-3}}{4\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\Gamma\left(\frac{1-s+i\tau}{2}\right)\Gamma\left(\frac{1-s-i\tau}{2}\right)\frac{\Gamma(s/2)}{(2-s)\Gamma((s-1)/2)}x^{-s}ds$$
$$=2^{-4}x^{-2}\int_0^x ydy\frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\Gamma\left(\frac{1-s+i\tau}{2}\right)\Gamma\left(\frac{1-s-i\tau}{2}\right)$$
$$\times\frac{\Gamma(s/2)}{\Gamma((s-1)/2)}y^{-s}ds\quad (x>0,\nu>0).$$

The value can be obtained by making use of the relation [4, §10, 9.55(4)]

$$(3.8)\qquad Y^{\psi}_{i\tau}(x)=-\frac{\sqrt{\pi}2^{-4}x^{-2}}{\cosh(\pi\tau/2)}\int_0^x y\frac{d}{dy}\left[K_{i\tau/2}\left(\frac{1}{y}\right)\left\{I_{-i\tau/2}\left(\frac{1}{y}\right)+I_{i\tau/2}\left(\frac{1}{y}\right)\right\}\right]dy,$$

where $I_\mu(z)$ is the modified Bessel function [1].

Theorem 9. *Let $0<\nu<1$ and $f(x)\in L_{\nu,p}(\mathbf{R}_+)$. Then under the condition $(\Phi f)(x)\in L_{1-\nu,1}(\mathbf{R}_+)$ for the operator (2.2) the inversion formula for the index transform (3.6) is given by*

$$(3.9)\qquad \int_0^x f(y)dy=-\underset{\varepsilon\to 0+}{\mathrm{l.i.m.}}\frac{1}{8\pi\sqrt{\pi}}\int_0^\infty\int_0^x\frac{\sinh((\pi-\varepsilon)\tau)}{\cosh(\pi\tau/2)}$$
$$\times y\frac{d}{dy}\left[K_{i\tau/2}\left(\frac{1}{y}\right)\left\{I_{-i\tau/2}\left(\frac{1}{y}\right)+I_{i\tau/2}\left(\frac{1}{y}\right)\right\}\right]g(\tau)dyd\tau,$$

where $g(\tau)$ in (3.6) is a bounded operator at any space $L_r(\mathbf{R}_+)$ $(r\geq 1)$ and the limit in (3.9) is meant by the norm of $L_{\nu-1,p}(\mathbf{R}_+)$. In addition, the limit in (3.9) exists almost everywhere on $\mathbf{R}_+$.

Example 3. The index transform with squares of the Bessel functions. We set $\varphi^*(s)=2^{s+1}/\{\Gamma(1-s/2)\Gamma((1+s)/2)\}$ in the formula (1.1). Evaluating the integral (1.2) by means of the formula [4, §10, 9.40(3)], we have the Mellin-Barnes representation

$$(3.10)\quad \frac{i\sqrt{\pi}}{x\sinh(\pi\tau/2)}\left[J^2_{i\tau/2}\left(\frac{1}{x}\right)-J^2_{-i\tau/2}\left(\frac{1}{x}\right)\right]$$
$$=\frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\frac{\Gamma((1-s+i\tau)/2)\Gamma((1-s-i\tau)/2)}{\Gamma(1-s/2)\Gamma((1+s)/2)}x^{-s}ds\quad (x>0)$$

for $0<\nu<1/2$, where $J_\mu(z)$ is the Bessel function of the first kind [1]. From (3.10) we obtain the new index transform with the squares of Bessel functions

$$(3.11)\qquad g(\tau)=\frac{i\tau\sqrt{\pi}}{\sinh(\pi\tau/2)}\int_0^\infty\left[J^2_{i\tau/2}\left(\frac{1}{y}\right)-J^2_{-i\tau/2}\left(\frac{1}{y}\right)\right]f(y)\frac{dy}{y}\quad (\tau>0).$$

The inversion kernel $Y_{i\tau}^{\psi}(x)$ for the transform (3.11) is evaluated in a similar way to the above, and by using relation [4, §10, 9.32(4)] we obtain

$$Y_{i\tau}^{\psi}(x) = \frac{\pi^{5/2}2^{-5}x^{-2}}{\cosh(\pi\tau/2)}\int_0^x y\frac{d}{dy}\left[J_{i\tau/2}^2\left(\frac{1}{y}\right) + Y_{i\tau/2}^2\left(\frac{1}{y}\right)\right]dy, \tag{3.12}$$

where $Y_\mu(z)$ is the Bessel function of the second kind [1].

Theorem 10. *Let $0 < \nu < 1/2$ and $f(x) \in L_{\nu,p}(\mathbf{R}_+)$. Then under the condition $(\Phi f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$ for the operator (2.2) the inversion formula for the transform (3.11) is given by*

$$\int_0^x f(y)dy = -\operatorname*{l.i.m.}_{\varepsilon\to 0+}\frac{\sqrt{\pi}}{16}\int_0^\infty\int_0^x \frac{\sinh((\pi-\varepsilon)\tau)}{\cosh(\pi\tau/2)} \times y\frac{d}{dy}\left[J_{-i\tau/2}^2\left(\frac{1}{y}\right) + Y_{i\tau/2}^2\left(\frac{1}{y}\right)\right]g(\tau)dyd\tau, \tag{3.13}$$

where $g(\tau)$ in (3.11) is a bounded operator at any space $L_r(\mathbf{R}_+)$ $(r \geq 1)$ and the limit in (3.13) is meant in the $L_{\nu-1,p}$-norm. Besides, the limit in (3.13) exists almost everywhere on $\mathbf{R}_+$.

Example 4. Index Re-Transform with squares of the Bessel functions. The final example of index transforms deals with the so-called Re-case of the previous index transform. This construction method of the index transform has been announced recently in [10], [12] and allows us to generalize effectively of index transforms of the Lebedev-Skalskaya type [12]. Putting in (1.2) $\varphi^*(s) = 2^{s+1}/\{s\Gamma((1-s)/2)\Gamma(s/2)\}$ and basing on the formula [4, §10, 9.40(3)], we can deduce the identity

$$\frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\frac{\Gamma((1-s+i\tau)/2)\Gamma((1-s-i\tau)/2)}{s\Gamma((1-s)/2)\Gamma(s/2)}x^{-s}ds = \frac{\sqrt{\pi}}{\cosh(\pi\tau/2)}\int_x^\infty \frac{1}{y^3}\operatorname{Re}\left[J_{-1/2-i\tau/2}^2\left(\frac{1}{y}\right) - J_{1/2+i\tau/2}^2\left(\frac{1}{y}\right)\right]dy \quad (x>0), \tag{3.14}$$

which is proved in the following, where Re means

$$\operatorname{Re}\left[J_{-1/2-i\tau/2}^2\left(\frac{1}{x}\right) - J_{1/2+i\tau/2}^2\left(\frac{1}{x}\right)\right] = \frac{1}{2}\left\{J_{-1/2-i\tau/2}^2\left(\frac{1}{x}\right) - J_{1/2+i\tau/2}^2\left(\frac{1}{x}\right)\right\} + \frac{1}{2}\left\{J_{-1/2+i\tau/2}^2\left(\frac{1}{x}\right) - J_{1/2-i\tau/2}^2\left(\frac{1}{x}\right)\right\}. \tag{3.15}$$

To prove the formula (3.14) and the representation for the inversion index kernel (see below), we use the elementary identities for the gamma-functions (see, for instance, [12]) as

$$\Gamma\left(a-b-\frac{1}{2}\right)\Gamma\left(a+b+\frac{1}{2}\right) + \Gamma\left(a+b-\frac{1}{2}\right)\Gamma\left(a-b+\frac{1}{2}\right) = (2a-1)\Gamma\left(a-b-\frac{1}{2}\right)\Gamma\left(a+b-\frac{1}{2}\right), \tag{3.16}$$

$$(3.17)\qquad \Gamma\left(a-b-\frac{1}{2}\right)\Gamma\left(a+b+\frac{1}{2}\right)-\Gamma\left(a+b-\frac{1}{2}\right)\Gamma\left(a-b+\frac{1}{2}\right)$$
$$=2b\Gamma\left(a-b-\frac{1}{2}\right)\Gamma\left(a+b-\frac{1}{2}\right)$$

for $a,b\in\mathbf{C}$. Let us prove (3.14). From the formula (1.2) we obtain

$$(3.18)\quad Y_{i\tau}^{\varphi}(x)$$
$$=\frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\frac{\Gamma((1-s+i\tau)/2)\,\Gamma((1-s-i\tau)/2)}{s\Gamma((1-s)/2)\Gamma(s/2)}x^{-s}ds$$
$$=\frac{1}{2\pi i}\lim_{M\to\infty}\int_{1-\nu-iM}^{1-\nu+iM}\frac{\Gamma((1-s+i\tau)/2)\,\Gamma((1-s-i\tau)/2)}{\Gamma((1-s)/2)\Gamma(s/2)}\int_x^{\infty}y^{-s-1}dyds.$$

In the last Mellin-Barnes integral the gamma-ratio has the order $O(|t|^{\nu-1/2})$ $(|t|\to\infty)$ for each $\tau>0$ and $s=1-\nu+it$. Changing the order of integration, we are led to the equality

$$(3.19)\quad Y_{i\tau}^{\varphi}(x)=\frac{1}{2\pi}\lim_{M\to\infty}\int_x^{\infty}y^{\nu-2}\int_{-M}^{M}\frac{\Gamma((\nu+i\tau-it)/2)\,\Gamma((\nu-i\tau-it)/2)}{\Gamma((\nu-it)/2)\Gamma((1-\nu+it)/2)}y^{-it}dtdy.$$

So if $0<\nu<1/2$ the inner integral by t in (3.19) converges boundedly, i.e. there exists a constant $C>0$ such that for any $M>0$ and $y>0$

$$\left|\int_{-M}^{M}\frac{\Gamma((\nu+i\tau-it)/2)\,\Gamma((\nu-i\tau-it)/2)}{\Gamma((\nu-it)/2)\Gamma((1-\nu+it)/2)}y^{-it}dt\right|\le C.$$

This fact follows from the Stirling formula for the gamma-function and the Slater theorem [4]. Passing to the limit by the Lebesgue theorem and using the relation [4, §10, 9.40(3)] and the identities (3.16) and (3.17), we arrive at (3.14).

Let us introduce the index transform with the kernel (3.14) as

$$(3.20)\; g(\tau)=\frac{\tau\sqrt{\pi}}{\cosh(\pi\tau/2)}\int_0^{\infty}\int_y^{\infty}\frac{1}{t^3}\mathrm{Re}\left[J_{-1/2-i\tau/2}^2\left(\frac{1}{t}\right)-J_{1/2+i\tau/2}^2\left(\frac{1}{t}\right)\right]f(y)dtdy$$
$$(\tau>0).$$

Changing variables in the double integral (3.20), we obtain the Re-transform

$$(3.21)\quad g(\tau)=\frac{\tau\sqrt{\pi}}{\cosh(\pi\tau/2)}\int_0^{\infty}\frac{1}{t^3}\mathrm{Re}\left[J_{-1/2-i\tau/2}^2\left(\frac{1}{t}\right)-J_{1/2+i\tau/2}^2\left(\frac{1}{t}\right)\right]f_1(t)dt\quad(\tau>0)$$

with respect to the function

$$f_1(t)=\int_0^t f(y)dy.$$

The inversion kernel $Y_{i\tau}^{\psi}(x)$ for the transform (3.11) is evaluated in the same manner as above by using the relation [4, §10, 9.32(4)] and the identity (3.17). So, we obtain

$$(3.22)\qquad Y_{i\tau}^{\psi}(x)=\frac{\pi^{5/2}2^{-3}x^{-2}}{\tau\sinh(\pi\tau/2)}\mathrm{Re}\left[J_{1/2+i\tau/2}^2\left(\frac{1}{x}\right)+Y_{1/2+i\tau/2}^2\left(\frac{1}{x}\right)\right].$$

Theorem 11. *Let $0 < \nu < 1/2$ and $f(x) \in L_{\nu,p}(\mathbf{R}_+)$. Then under the condition $(\Phi f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$ for the operator (2.2) the inversion of the index transform (3.20) is given by*

$$f_1(x) = \underset{\varepsilon\to 0+}{\mathrm{l.i.m.}} \frac{\sqrt{\pi}}{4}\int_0^\infty \frac{\sinh((\pi-\varepsilon)\tau)}{\tau\sinh(\pi\tau/2)} \mathrm{Re}\left[J^2_{1/2+i\tau/2}\left(\frac{1}{x}\right) + Y^2_{1/2+i\tau/2}\left(\frac{1}{x}\right)\right] g(\tau)d\tau, \tag{3.23}$$

where

$$f_1(x) = \int_0^x f(y)dy$$

and $g(\tau)$ in (3.20) is a bounded operator in the space $L_r(\mathbf{R}_+)$ $(r \geq 1)$. Moreover the limit in (3.23) is meant in the $L_{\nu-1,p}$-norm. Besides, the limit in (3.23) exists almost everywhere on $\mathbf{R}_+$.

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